A flexible approach to matrix balancing under partial information

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Abstract

We present an extension of the standard RAS algorithm, referred to as 'constrained RAS', or 'cRAS'. In contrast to existing RAS variants employing additional partial information in a particular aggregated form, our technique makes use of partial information of any kind. This is achieved by imposing constraints on arbitrary-sized and –shaped sets of elements to be balanced. The characteristics of the cRAS method are investigated in detail.

1 Introduction

A common problem in compiling and updating input-output tables is that of incomplete data. Missing matrix elements may be due to a variety of reasons such as costly and therefore incomplete industry surveys, or the suppression of confidential information. Unknowns almost always outnumber data points, resulting in the system being underdetermined, that is exhibiting too many degrees of freedom to be solved analytically, so that certain estimation, balancing or optimisation techniques have to be applied.

Despite having received mixed responses in relation to the accuracy of its results[1], a widely applied remedy in this situation is *the bi-proportional*, or RAS method (Stone and Brown 1962). Especially for the task of constructing or updating regional input-output tables from national data, the RAS method has been compared with other approaches such as location quotients and commodity balances (Schaffer and Chu 1969; Leontief 1953), gravity models (Leontief and Strout 1963), linear programming (Moses 1960; Matuszewski *et al*. 1964; Davis *et al*. 1977), or entropy and minimum information gain techniques (Uribe *et al*. 1966; Batten 1982; Golan *et al*. 1994). The origins of the method go back several decades (Deming and Stephan 1940); for a historical background, see Bacharach 1970, Chap. 1-3, and Polenske 1997.

The RAS method – in its basic form – bi-proportionally scales a matrix A_0 of un-

balanced preliminary estimates of an unknown real matrix **A**, using **A**'s known row and column sums. The balancing process is usually aborted when the discrepancy be- tween the row and column sums of $\mathbf{A}_0$ and **A** is less than a previously fixed threshold. Bacharach 1970 has analysed the bi-proportional constrained matrix problem in great detail, in particular in regard to the economic meaning of bi-proportional change[2], the existence and uniqueness of the iterative RAS solution, its properties of minimisation of distance metric[3], and uncertainty associated with errors in row and column sum data and with the assumption of bi-proportionality.

A special situation arises when some of the matrix elements of **A** are known in addition to its row and column sums, for example from an industry survey. The 'modi- fied RAS' approach (Paelinck and Walbroeck 1963; Allen 1974; Lecomber 1975 deals with this partial information as follows: the preliminary estimate $\mathbf{A}_0$ has to be "netted", that is the known elements are subtracted, and $\mathbf{A}_0$ contains 0 at the corresponding posi- tions. The net $\mathbf{A}_0$ is then subjected to the standard RAS procedure, and the known ele- ments are added back on after balancing. An operational example of this approach is the GRIT (Generation of Regional Input-output Tables) method (Jensen *et al*. 1979; West *et al*. 1984; Bayne and West 1989), which involves 4 steps: 1) preparation of the national table, 2) preliminary estimate of the regional table using a non-survey ap- proach such as commodity balances or location quotients[4], 3) insertion of superior, survey-based data, and 4) identification of important elements, and subsequent collec- tion and insertion of superior data. Within GRIT, aggregated superior data can only be inserted after aggregating the preliminary table (Jensen *et al*. 1979, Sec. 6.4). Another variant of the modified RAS method takes into account the uncertainty of the prelimi- nary estimates, and contains the occurrence of perfectly known elements as a special case[5] (Allen and Lecomber 1975).

In practice, situations can arise where, in addition to certain elements of **A**, some

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[1]   See for example Uribe *et al*. 1966; Czamanski and Malizia 1969; Goldman 1969; Miernyk 1969; Malizia and Bond 1974; Morrison and Smith 1974; Miernyk 1976; Hinojosa 1978; Parikh 1979; Butterfield and Mules 1980; Morrison and Thuman 1980. Jensen *et al*. 1979 summarise the dilemma of regional input–output modeling as a trade-off between 'professional purity' and uncertain accu- racy.

[2]   When applied to the forecasting of monetary input–output matrices, bi–proportional changes have been interpreted as productivity, substitution or fabrication effects (Leontief 1941; Stone and Brown 1962) affecting industries over time. Miernyk's (1976) view however is that the RAS method "substi- tutes computational tractability for economic logic", and that the production interpretation loses its meaning when the entire input–output table is balanced, and not only inter–industry transactions (see also Giarratani 1975).

[3]   The RAS, Linear Programming and minimum information gain algorithms yield a balanced matrix estimate that is − in terms of some measure of multidimensional 'distance' − closest to the unbal- anced preliminary estimate. When applied to temporal forecasting, this property is explained as a conservative hypothesis of attributing inertia to inter–industrial relations (Bacharach 1970, p. 26).

[4]   For an overview of these techniques see Schaffer and Chu 1969, or Harrigan *et al*. 1980.

[5]   This is accomplished by introducing a matrix **E** of standard errors of the elements in $\mathbf{A}_0$, which is subsequently balanced in order to take up the difference between the preliminary and true totals. Where $E_{ij}=0$, the element in $\mathbf{A}_0$ remains unchanged. Allen and Lecomber 1975 also investigate the influence of errors in the "true" totals.

aggregates of elements of **A** are known. For example, a published table $\mathbf{A}^G$ of national aggregates may constitute partial information when constructing a multi-regional input-output system, or a more disaggregated national table. Accordingly, Oosterhaven *et al.* 1986 add a "national cell constraint" to the standard row and column sum constraints. Similarly, Jackson and Comer 1993 use partition coefficients for groups of cells of a disaggregated base year matrix to disaggregate cells in an updated but aggregated matrix. Batten and Martellato 1985, p. 52-55) discuss further constraints structures, involving intermediate and final demand data. Gilchrist and St Louis 1999 propose a three-stage RAS for the case when aggregation rules exist under which the partial aggregated information $\mathbf{A}^G$ can be constructed from its disaggregated form A. These rules take the form of matrix operators **P** and **Q**, with $\mathbf{A}^G = \mathbf{PAQ}$. The essence of their TRAS method is that balancing is carried out alternately at the disaggregated and at the aggregated level, with the aggregated row and column scaling operators $\mathbf{R}^G$ and $\mathbf{S}^G$ being disaggregated via multiplication with the transposed operators $\mathbf{P}^t$ and $\mathbf{Q}^t$. Subjecting an input-output matrix to random censoring, Gilchrist and St Louis 1999 demonstrate that the inclusion of partial aggregated information into the RAS procedure leads to superior outcomes than applying the standard RAS method.

In the above approaches the partial information has to assume a particular aggregated form. A number of authors[6] have expressed the table estimation as an econometric or a linear or quadratic programming problem, which allows the balancing of negative elements and the inclusion of partial information of any form. Lahr and de Mesnard generalised the method 1) by formulating it as a Lagrangian multiplier calculus, as an econometric estimation, or as an optimisation, 2) by enabling balancing of negative elements, and 3) by allowing for partial information of any form, by imposing constraints on arbitrary-sized and –shaped sets of elements of $\mathbf{A}_0$. Lahr and de Mesnard 2004 provide a recent overview.

The aim of this work is to test the performance of approaches that feature arbitrary constraints. In particular we investigate whether
   1) it is better to constrain a few large elements, or many small elements,
   2) it is better to have a few constraints spanning many elements, or many constraints on single elements,
   3) the shape of the constraints (adjacent elements or disjunct elements) influences the performance.
We enumerate performance using a range of common distance measures, and apply them to the flow matrix, the Leontief inverse, and some multipliers. We restrict ourselves to testing a RAS-type approach (referred to as 'constrained RAS', or 'cRAS'), and ignore Lagrangian and optimisation approaches.

This article is organised as follows: We describe our proposed cRAS method, then present an application to a generalised regional input-output system for Australia, and conclude.

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[6] See for example Byron 1978, Morrison and Thuman 1980, van der Ploeg 1982 and Cole 1992 on the Lagrangean method, Gerking 1976; 1979a and Toh 1998 on econometric estimation, and Tarancon and Del Rio 2005 on optimisation. See Junius and Oosterhaven 2003 and Jackson and Murray 2004 on the problem of negative entries. See Morrison and Thuman 1980, Cole 1992, Thissen and Löfgren 1998 and Tarancon and Del Rio 2005 on generalized constraints.

2 A representative cRAS method

In the standard RAS method, the preliminary estimate A_0 is alternately row- and column-scaled using diagonal matrices $\hat{r}$ and $\hat{s}$, so that after one round of balancing,. $A_1 = \hat{r}A_0\hat{s}$. A_1 is then subjected to the next scaling operation. One may start with either rows, $A_1 = (\hat{r}A_0)\hat{s}$, or columns, $A_1 = \hat{r}(A_0\hat{s})$. In the first round, $\hat{r}_{ii} = a_i / \sum_j A_{0,ij}$ and $\hat{s}_{jj} = a_j / \sum_i A_{0,ij}$, and so on, where the a_{ij} are row/column totals of A.

Let C be the number of additional available data points to be used as constraints on the A_{ij}. Let E_c be the number of elements of A participating in constraint c. Let $(i_{c,e}, j_{c,e})$ be the coordinates of element e in constraint c. Let $B_c = \sum\limits_{e=1}^{E_c} A_{i_{c,e}\,j_{c,e}}$ be the value prescribed for the sum of elements participating in constraint c. A typical cRAS algorithm can then be described as a sequence:

- Generate preliminary estimate A_0;
- scale rows of A_0 by pre-multiplication with $\hat{r}$;
- scale constraint 1 by multiplying all E_1 participating elements with

$$B_1 \Big/ \sum_{e=1}^{E_c} A_{0,i_{1,e}\,j_{1,e}} = \sum_{e=1}^{E_c} A_{i_{1,e}\,j_{1,e}} \Big/ \sum_{e=1}^{E_c} A_{0,i_{1,e}\,j_{1,e}} \,;$$

- scale constraint 2 by multiplying all E_2 participating elements with ;

$$B_2 \Big/ \sum_{e=1}^{E_c} A_{0,i_{2,e}\,j_{2,e}} = \sum_{e=1}^{E_c} A_{i_{2,e}\,j_{2,e}} \Big/ \sum_{e=1}^{E_c} A_{0,i_{2,e}\,j_{2,e}} \,;$$

- and so on, until constraint C;
- scale columns by post-multiplication with $\hat{s}$;
- result of the above operations is A_1;
- repeat scaling process for A_1 until all scaling factors deviate from 1 by less than a specified threshold.

Any permutation of this sequence is also a valid cRAS algorithm. Constraints can include any number of elements, which may be fully, partly or non-adjacent[7]. Constraints may also exclude some of the row and column totals (compare Thissen and Löfgren 1998, p. 1994).

3 Performance evaluation

In the following, we will examine the performance of the cRAS method for the example of estimating the use matrix U of the 1996-97 Australian input-output tables from its true row and column sums, as well as varying numbers of subsets of true elements,

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[7] Constraints with $E_c = 1$ need not be part of the scaling procedure, but could be "netted out" using the "modified RAS" method.

starting with different preliminary estimates. We will also appraise estimates for the Leontief inverse $L = (I - U\hat{x}^{-1})^{-1}$, and output and CO_2 multipliers $\xi = 1L$ and $\chi = cL$, respectively, where I is a unity matrix, $\hat{x}$ is the diagonalised gross output vector, 1 is a unity row vector, and c is a row vector of sectoral CO_2 intensities.

While a performance evaluation based on U emphasises the importance of the table elements themselves, evaluations based on L, ξ and χ recognise that these derived quantities are ultimately of interest in applied models. With each subsequent step of further processing U (through inversion to L, summing to ξ and weighting to χ), the derived quantities embody more widely inter-linked information. Evaluations of these quantities (rather than table elements) are the subject of a more integrated and applied measure of overall, 'holistic', or operational accuracy (Jensen 1980). While table accuracy represents the conventional understanding of a measure of the deviation of all estimated matrix elements from their "true" values, holistic accuracy is concerned with the representativeness of a table of the synergistic characteristics of an economy. In this perspective, the accuracy of single elements may be unimportant, as long as the results of modeling exercises yield a realistic picture for the purpose of the analyst or decision-maker. Table accuracy implies holistic accuracy, but not vice versa; nevertheless, the former may not be achievable, while the latter may be more realistic in pragmatic terms. However, a drawback is that general criteria for holistic accuracy are hard to define, and hence are a characteristic of the model output, such as the derived quantities above.

3.1 Programming features

A number of stochastic experiments of up to 250 runs were carried out, in which the preliminary estimate, the number of constraints, the numbers E_c and coordinates $(i_{c,e}, j_{c,e})$ of constrained subset elements, and other parameters were varied.

3.1.1 Balancing threshold
It is in principle possible to terminate the balancing process based on absolute or relative criteria. For example, a threshold can be formulated as $\max_i (|u_i - u_{n\,i}| / u_i)$, where u contains the row and column sums of U, and u_n those of the nth iteration of U_0. However, industry sectors would receive equal weights independent of their size. Therefore, we adopted the criterion $|u_i - u_{n\,i}| < \overline{U}a$, for the row and column totals, and also for all additional constraints, where $\overline{U} = N^{-2} \sum_{ij} U_{ij}$ is the average absolute value of the true flow matrix, and a is a scaling factor. This way, the balancing result will exhibit an overall absolute proximity to constraints, which is better by a certain factor than the average magnitude of the matrix itself.

3.1.2 Preliminary estimate
We allowed three different types of preliminary estimates U_0: 1) the 1994-95 Australian flow matrix U^{95}, 2) a weighting $U_{ij}^1 = (u_i\, u_j) / \Sigma_{mn} U_{mn}$ according to industry size, and 3) random numbers U_{ij}^R with mean $\overline{U}$. Option 1 represents a classical updating task. Option 2 resembles an estimation task (often of regional tables) in the absence of any data except prescribed row and column sums; the formulation is the most simple prediction for inter-industry flows. It is based on the assumption of independence, that is it im-

Fig. 1: Constraint structures and coordinate locations in the flow matrix U, for the example of distributing 50 constraints with 20 elements each. Set elements can be either disjunct (left) or adjacent (right). Their coordinates can assume either random positions (top) or be skewed towards top-ranking elements (bottom). Note that in the numerical experiments the number of set elements does not have to be constant.
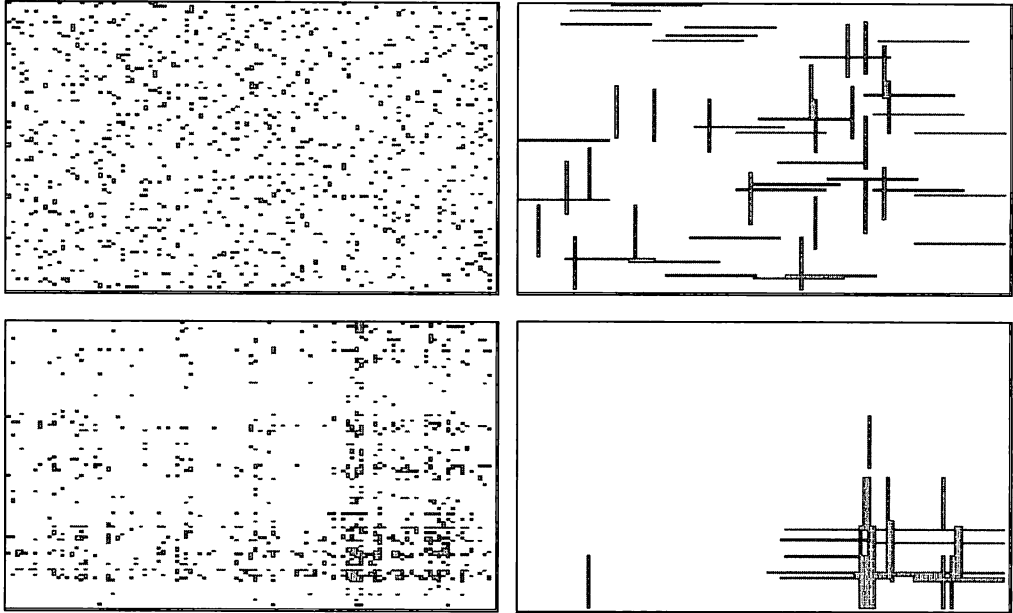

plies that flows from industry i to industry j are large if i produces much and j uses much. Since the production recipe, transport distances or other parameters do not play any role, this estimate is generally quite unrealistic (Uribe *et al.* 1966). Note that in the absence of additional subset constraints, option 2 already satisfies row and column constraints, since $\Sigma_i \, (u_i \, u_j) \, / \, \Sigma_{mn} \, U_{mn} = u_j$, and therefore cannot precede simple RAS balancing.

3.1.3 *Constraint size and structure*
In principle, the elements belonging to any particular constraint can be an arbitrary-shaped and –sized subset of the elements of U. This was simulated by randomly selecting the number and size of constraints as well as the coordinates of elements. The constraint value is the sum over values of the target matrix elements that belong to the respective constraint. In this work, all constraints are consistent by definition, because their values are determined from the 1996-97 U matrix.

The elements belonging to any constraint can be anything from completely disjunct (referring to completely different pairs of industries) to all adjacent (having industries in common). One common example for completely disjunct constraints are so-called "rectangular" constraints which arise out of constraining all elements U_{ij}^{rs} connecting industries i and j and regions r and s of a multi-regional input-output table, by the national estimate U_{ij}. More random disjunct constraints can occur if a table expressed in a certain industry sector classification is constrained using data expressed in

Fig. 2: Number of constrained elements ($\triangle$, left y-axis) and overlap / duplication ratio ($\bigcirc$, right y-axis) as a function of the number of set elements (all constraint structures, equal selection of constraint element coordinates).
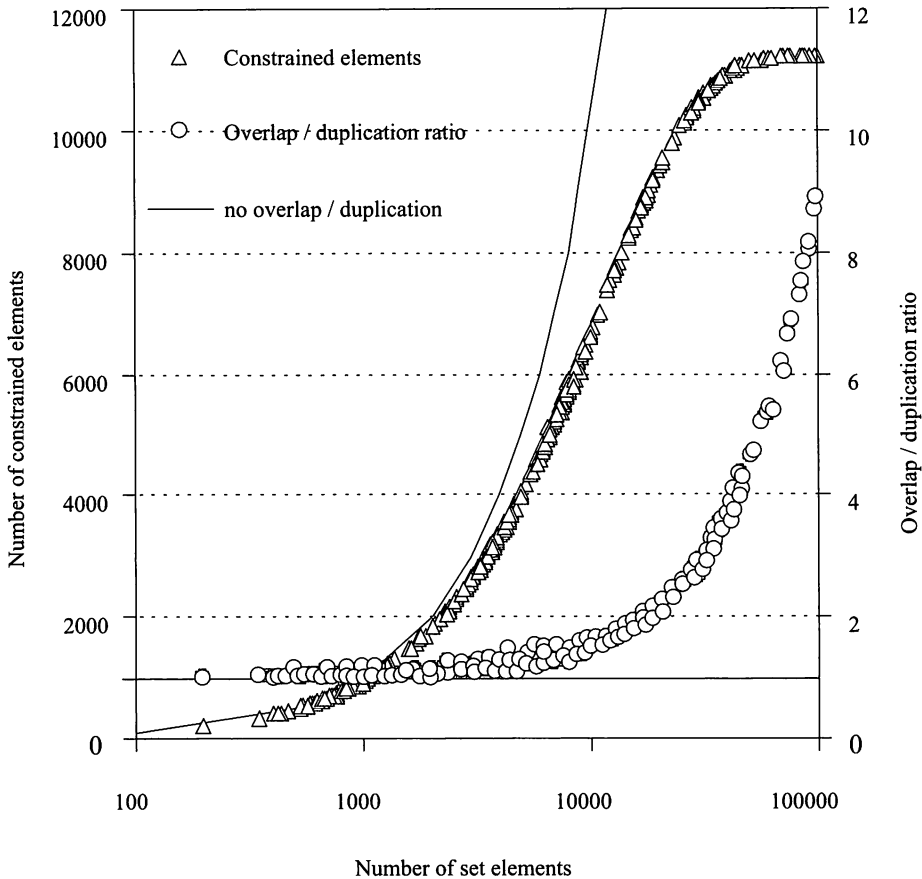
another industry sector classification. In Australia, for example, superior data often exists classified as ANZSIC[8], while input-output tables are classified in IOPC[9]. The most common versions of these, ANZSIC 4-digit and IOPC 4-digit, overlap in a sense that for some parts of the economy, ANZSIC is more detailed, for others IOPC. Moreover, while IOPC is structured to minimise the output of secondary products for all sectors, ANZSIC could include entire establishments with multiple output commodities in one sector[10]. The result of constraining an IOPC system with ANZSIC data are constraints that are scattered over non-adjoining rows as well as columns (Fig. 1 left).

It is equally likely to encounter constraints that incorporate adjacent elements, which mostly result from aggregated superior information, for example the intermediate input of a whole group of commodities, or the output of a whole group of industries.

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[8] Australian and New Zealand Standard Industry Classification.

[9] Input–Output Product Classification.

[10] An example is broadacre farming, where combined beef–wheat farms are classified in one ANZSIC sector, but split into two IOPC sectors.

Fig. 3: Probabilities for selecting elements from a ranked list, obtained from 2000 runs. The curve parameter is the number r of randomisations. $r = 1$ selects all ranks equally.
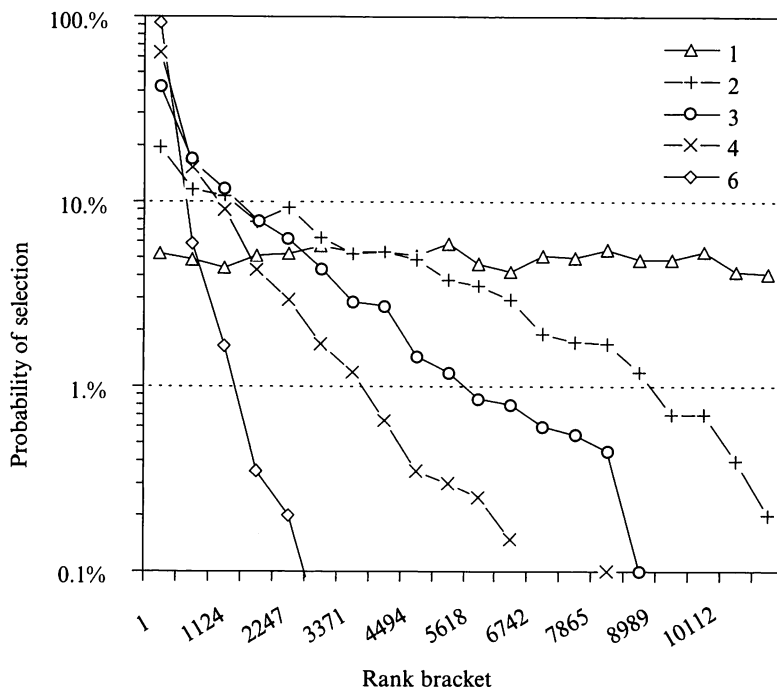
These will appear as either vertical (inputs) or horizontal (outputs) bands of the table to be estimated (Fig. 1 right).

In general, subsets of randomly selected elements will be mostly disjunct. In order to generate adjacent-cell subsets, a seed element was first selected randomly, and then only adjacent cells were selected for the remaining elements up to the respective constraint size. In this work, only row- and column-segment-shaped constraints (with random orientation) were considered as adjacent-cell subsets.

With increasing number and/or size of constraints, overlaps and/or duplications become more likely. In the following we refer to the (gross) number of elements in all constraints as 'set elements', while a smaller number of elements (net of overlap) are actually 'constrained'. If set elements are few, they are likely not overlapping (Fig. 2, set elements = constrained elements, triangles and solid line coincide; overlap ratio = 1). As the number of set elements increases, overlap/duplication increases: the number of actually constrained elements falls below the number of set elements contained in all constraints (Fig. 2, set elements > constrained elements, triangles and solid line deviate; overlap ratio >1).

3.1.4 *Coordinate location*

In principle, constraints can incorporate arbitrary elements, or focus on important elements. In order to cover these alternatives in our simulations, set elements (Fig. 1 left) or seed elements (right) were either selected equally (top) or preferentially (bottom)

Fig. 4: Distribution of element numbers and values in the 1994/95 Australian flow matrix across size brackets. While there are large numbers of elements near zero and between A\$1m and A\$100m, most of the total intermediate transaction value is concentrated in a few elements of A\$1b and larger.

Flow value bracket log(U_{ij} \$)
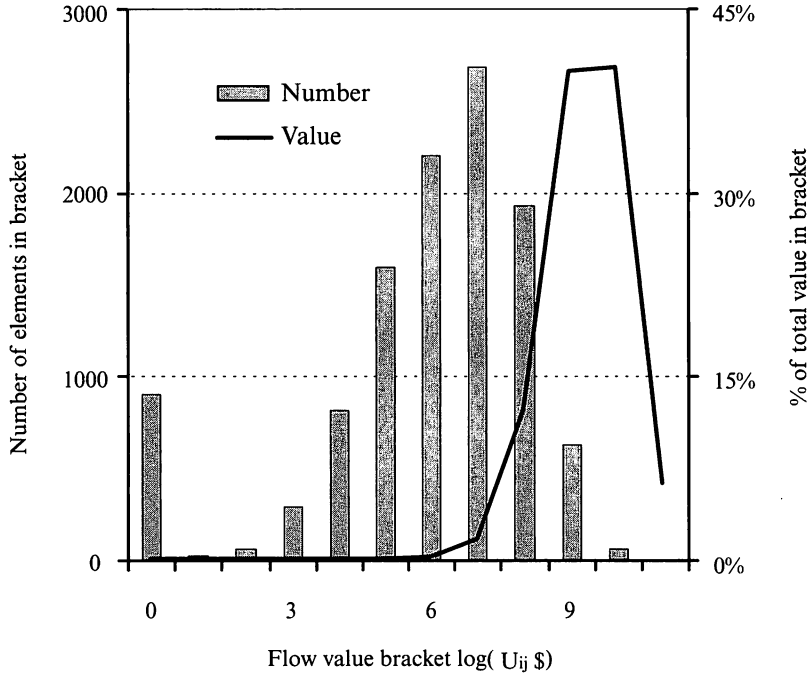
from a ranking of important elements. The latter were defined as those elements that have the largest value in the preliminary estimates U^{95} or U^1.

Preferential selection (Fig. 1 bottom) can be realised by generating random rank numbers (in this case from 1 to $N^2 = 106^2$) with a larger probability for lower numbers (upper ranks) in order to ensure that these are always part of a constraint subset. Such a probability distribution can be constructed by repeatedly applying a random number $Rand \in [0,1[$ as the upper bound of a subsequent random number. An algorithm that carries out this task is quite simple:

$x = 1.$: For $k = 1$ To r : $x = x \times Rand$ ([0,1[) :Next k : $rank = x \times N^2$

The mean of random values x generated in such a way decreases with the number of randomisations r. Based on once-randomised ($r = 1$) values in [0,1[, elements of *rank* rank are selected with equal probability (curve 1). Since low numbers $x \in [0,x[$ occur with increasing frequency after repeated randomisations ($r > 1$), top-ranking elements are selected with increasing probability (Fig. 3).

A streamlined constraint selection is much in the spirit of 'GRIT' and similar techniques: Jensen and West 1980 report that a surprisingly large number of smaller elements in a flow matrix can even be removed before output multipliers show a significant change, because the value of these elements is often negligible compared to the combined value of a few large flow values (see Fig. 4). Hewings and Jensen (1989, p. 314) note that not necessarily the largest elements are the most important in terms of

their effect on multipliers, but that both coefficient size and location matter. Jensen and West 1980, West 1981 and Hewings 1984 propose generating a ranking of matrix elements in terms of their effect on multipliers, in order to identify elements for which more accurate information should be collected. These considerations could determine selection procedures similar to the simple one chosen here.

3.1.5 Distance measures

The results of the balancing process are compared with the published 1996-97 matrices. A number of authors[11] examine concepts of relative distance between two matrices in order to characterise the comparative performance of matrix balancing methods. According to Butterfield and Mules (1980, p. 293), "there exists no single statistical test for assessing the accuracy with which one matrix corresponds to another. Analysts working in this area have tended to use a number of [complementary] tests." Accordingly, we chose seven measures (compare Harrigan et al. 1980, Günlük-Senesen and Bates 1988, and Lahr 2001):

- the relative arithmetic mean of absolute differences $AMAD = \dfrac{\sum\limits_{ij} |U_{ij} - U_{nij}|}{\sum\limits_{ij} U_{ij}}$;

- the relative geometric mean of absolute differences[12] $GMAD = \dfrac{\sqrt{\sum\limits_{ij} |U_{ij} - U_{nij}|^2}}{\sqrt{\sum\limits_{ij} U_{ij}{}^2}}$;

- the Isard/Romanoff Similarity Index[13] $SIM = 1 - \dfrac{\sum\limits_{ij} \dfrac{|U_{ij} + U_{nij}|}{U_{ij} + U_{nij}}}{N^2}$;

- the χ^2 distribution of absolute differences $CHI = \sum\limits_{ij} \dfrac{(U_{ij} - U_{nij})^2}{U_{ij}}$;

- the arithmetic mean of relative differences $AMRD = \dfrac{\sum\limits_{ij} \dfrac{|U_{ij} - U_{nij}|}{U_{ij}}}{N^2}$;

- the information index $INFO = \sum\limits_{ij} U_{nij} \log_2 \dfrac{U_{nij}}{U_{ij}}$; and

- the correlation coefficient $CORR = \dfrac{\mathrm{Cov}\,(U_{ij}, U_{nij})}{\sqrt{\mathrm{Var}\,(U_{ij})}\sqrt{\mathrm{Var}\,(U_{nij})}}$,

where $\mathbf{U}_n$ is the nth iteration of $\mathbf{U}_0$, and $N = 106$ the dimension of $\mathbf{U}$ and the $\mathbf{U}_n$.

3.1.6 Stochastic experiments

In the following, each experiment will be characterised by a selection of parameters, such as

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[11] Czamanski and Malizia 1969; McMenamin and Haring 1974; Morrison and Smith 1974; Lecomber 1975; Harrigan et al. 1980; Jackson and Comer 1993

[12] The square of this measure has also been referred to as 'Theil's inequality index' (Lahiri 1984).

[13] 1 − SIM has been used as a "Dissimilarity Index" by Thissen and Lofgren 1998.

- the preliminary estimate: 1994/95 flow matrix U^{95}, independent flows U^{I}, random numbers U^{R};
- the constraint structure: disjunct, adjacent;
- the selection of constrained elements, r: 1: equally, 2 and 3: increasing degree of preference for top-ranking flows;
- constraint size: number of elements contained in constraints;
- total $ value of set elements: percentage of total $ value of flow matrix U;
- number of constraints;
- accuracy type (table $\rightarrow$ holistic): distance test performed on flow matrix U, Leontief inverse L, output multipliers ξ, CO_2 multipliers χ.

4 Results

The following results are a selection from thousands of balancing runs carried out on a contemporary desktop computer, programmed in simple Visual Basic. While in most runs the threshold was reached after 30 seconds or so, only a minority of the runs took up to 5 minutes computing time.

4.1 Influence of the balancing threshold

In order to illustrate the effect of varying the balancing threshold[14], we have chosen an example with a potentially long balancing procedure, where an initially unrealistic estimate (U^{I}) is substantially altered during balancing, based on a large number (2500) of additional data points. All distance measures reach a stationary state already when the balance criterion $a_n = |u_i - u_{ni}| / \overline{U}$ falls below 10 (Fig. 5, left graph, $\triangle$). This may seem surprisingly large, however considering the size distribution of elements (Fig. 4), 10 $\overline{U}$ is in the order of $10^7A\$$, which is well below the value range that contains the bulk of the total transaction value.

As expected the correlation and similarity indices increase, while absolute differences and the information gap decrease (Fig. 5). Interestingly, the subsequent row and column balance improves the balance criterion (grey parts of $\triangle$, left graph), but counteracts the distance measures by unbalancing the constraint subsets (black parts). This alternating process of constraint and row/column balancing causes all performance measures to perform a damped oscillation.

The absolute mean of relative differences (AMRD) is much higher then the absolute mean of absolute differences (AMAD), since the former is influenced by a large number of small elements that are not well approximated. Moreover, the AMRD as well as the χ^2 and information measures do not seem to recover from the effect of the second row/column balancing just after step 2500. We could not understand and explain this feature in detail, but re-examining the formulae in Sec. 1.1.5 shows that these

[14] Polenske (1997, p. 79) already pointed out that "analysts should consider [... the issue of the] minimum level of discrepancy allowed between control totals and estimated totals", and that "information concerning these issues is so limited".

Fig. 5: Convergence of a typical cRAS run for the example of "initial estimate UI, 2500 disjunct constraints of size 1 and total $ value 50%U, selected with $r=$ 2 preference, distance test on U". The curves show a record of the balance criterion $a_n = |u_i - u_{ni}|/\overline{U}$, and all distance measures. The intermittent steps (some visible in grey) show the log of row and column adjustments (counted as 1 step for all rows, and 1 step for all columns), while the remaining black parts represent the adjustments (1 step each) of the additional constraints.
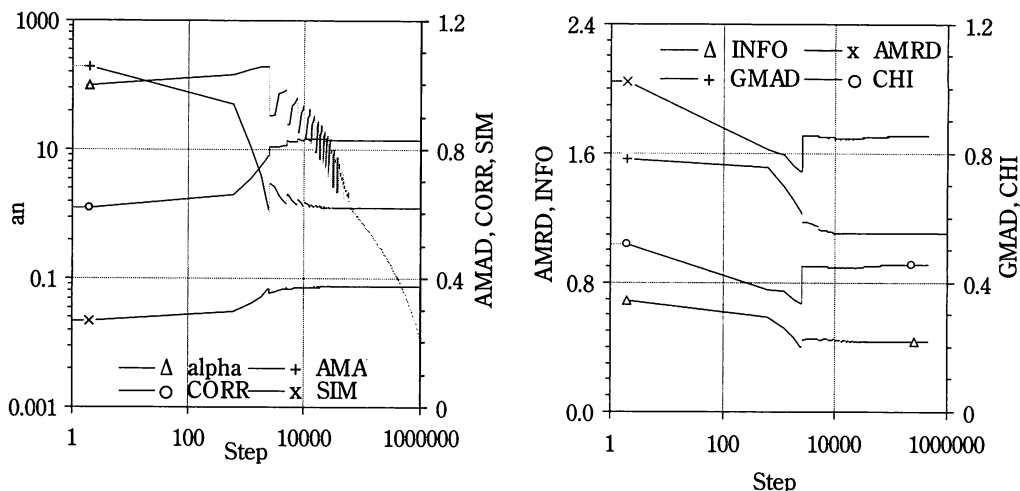

measures are the only ones that feature a single term U_{ij} in the denominator, and not a sum. It appears that because of this characteristic, these three measures are quite susceptible to suboptimal balancing outcomes of small (and potentially unimportant) elements, which was confirmed in further simulations. The GMAD behaves very similarly to the AMAD, but varies less. Similarly, CORR and SIM vary less over a single and between runs. The above results were confirmed when the constraint structure, size, numbers and value were changed, and as a consequence, the AMAD was chosen as the preferred performance measure in most of the analyses below.

4.2 Influence of the preliminary estimate

The importance of the preliminary (pre-RAS) estimate is well known from experiences in the estimation of regional input-output tables, and our results confirm this. If the 1994-95 flow matrix U^{95} is used as a start matrix in an updating exercise, the resulting balanced estimate is much closer to the 1996-97 flow table U^{97} (Fig. 6, left diagram) than in the case of the independence estimate U^{I}: Similarity and AMAD are about 0.2 points up, and 0.6 points down, respectively, and the correlation is close to unity. At the other extreme, a random matrix with mean $\overline{U}$ (Fig. 6, right diagram) is quite a bad approximation, but this cRAS outcome performs similarly to that resulting from the independence estimate (Fig. 5), even though the industry-size-weighted form makes more intuitive sense (compare Hewings 1977, p. 940). Varying the additional constraints yielded table accuracy AMADs of around 95%, 85% and 20%, for U^{R}, U^{I}, and U^{95} respectively. This finding confirms Polenske's (1997) review[15], and also Miernyk's (1969;

Fig. 6: Convergence of cRAS for the example in Fig. 5, except initial estimates U^{95} (left diagram) and U^R (right diagram). The curves show a record of the balance criterion $a_n = |u_i - u_{ni}|/\overline{U}$, and the distance measures AMAD, CORR and SIM. The intermittent steps (some visible in grey) show the log of row and column adjustments (counted as 1 step), while the remaining black parts represent the adjustments (1 step each) of the additional constraints.
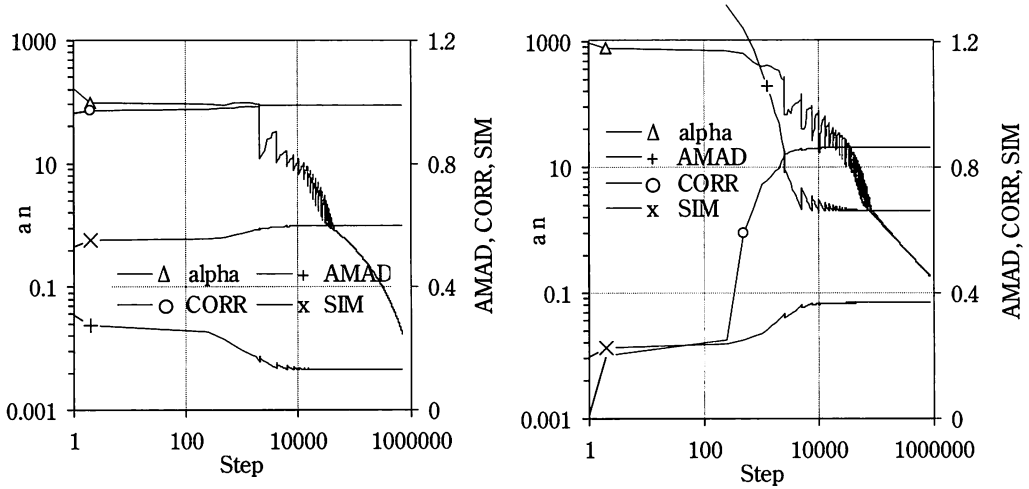
1976) view that – at least in terms of table accuracy – regional input-output estimates can be an inadequate substitute for real data. In all cases the main impetus towards improving table accuracy are the additional constraints, and not the row and column constraints. Additional constraints carry much more information about U^{97} than both U^I and U^R, making them appear almost equal in comparison.

4.3 Influence of the constraint selection and size

In practice, the analyst is most likely to be confronted with finding data for a certain number of constraints, while limited in time and resources. In looking for superior information, the analyst can choose between constraints of different size and $ value, subject of course to data availability. A crucial question is then which constraints would lead to good results in terms of approximating the target matrix: a few large constraints spanning many elements, or many small constraints? Is it efficient to focus on potentially large elements? In the following we investigate these issues by varying constraint selection and size. We test for disjunct structures only, since results for adjacent structures are quite similar (see also Sec. 1.4).

Generally speaking, adding more constraints improves table accuracy, provided the

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Fig. 7: Performance of cRAS runs for the example of "initial estimate U^I, N disjunct constraints of varying size and undefined total $ value, selected with $r=1,2,3$ preference, distance test on U". Increasing the number of constraints has a larger impact on accuracy of U than increasing constraint size or selectivity.
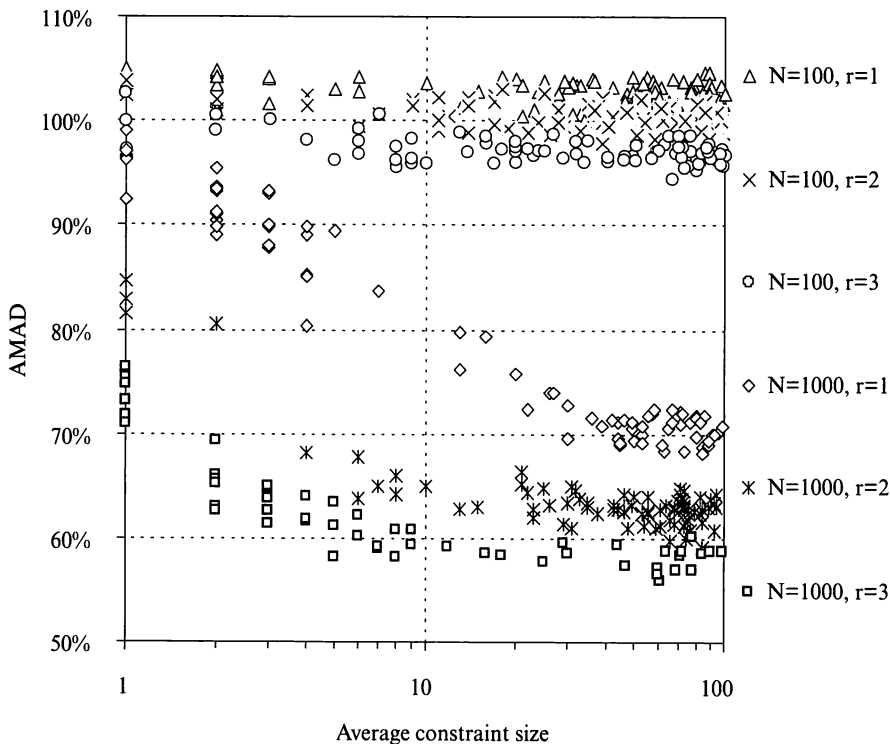
data are not conflicting. In our example, an increase in the number of constraints from 100 to 1000 led to a 30-40% decrease in the AMAD (Fig. 7, compare curves △×○ for 100 constraints with curves ◇＊□ for 1000 constraints). In the case of 100 constraints, altering the constraint size (1 – 100 elements) or selection (equally $r = 1$, increasing focus on large transactions $r = 2, 3$, see Fig. 3) did not change the AMAD significantly. The first finding is remarkable since as it says that for the estimation accuracy to be expected from 100 constraints, it does not matter much whether these contain 100 or 10,000 set elements.

With increasing number of constraints, the way of selection and their size become important (compare Hewings and Janson 1980, p. 848; Lynch 1986, pp. 280-282). First, focusing on large transactions is always benefitial, although $r = 3$ produces only slightly better results than $r = 2$. Constraint size increases are also benefitial, however exceeding 10 elements improves accuracy only marginally, at least for preferential selection. The declining impact of the preferential selection is due to the fact that already at $r = 2$, almost all top elements have been selected. The declining impact of constraint size increase can be understood in the sense that the information contained in larger constraints is less specific to potentially important single elements within the constraint, and hence table accuracy is diminished.

Fig. 8: Performance of cRAS runs for the example in Fig. 7, but with varying total constraint $ value, and undefined size. Compared to covering more $-value with constraints, selectivity makes a minor difference to accuracy of U.
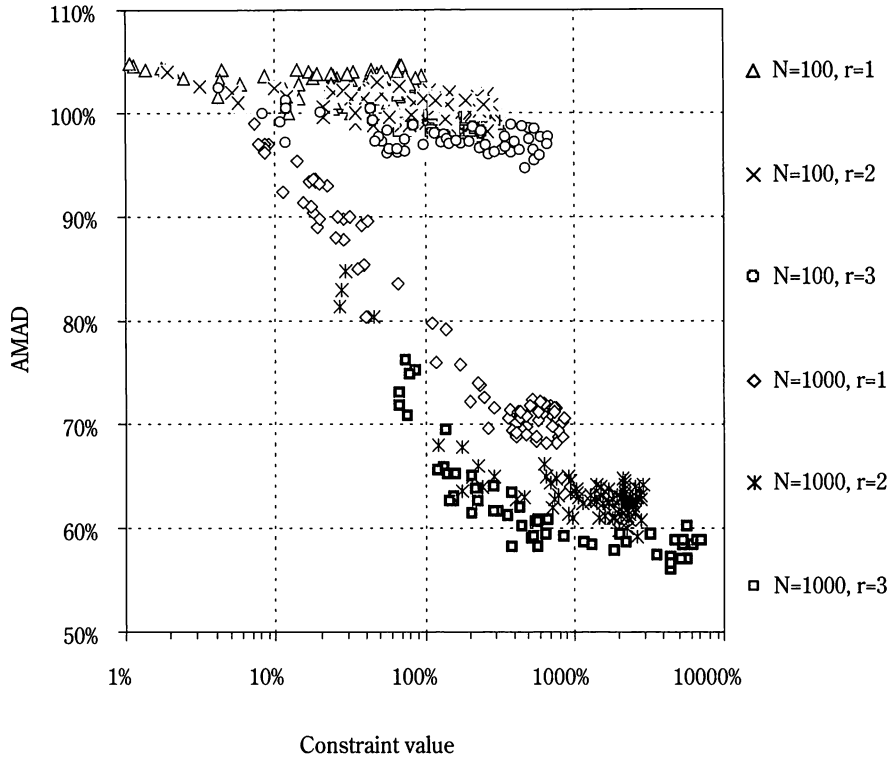

Constraint value

Rather than constraint size and hence number of set elements, the total transaction value covered by the constraints appears to be a more important determinant (Fig. 8). 100 constraints can potentially cover 100% of the transaction value (and more for preferential selection, due to overlap/duplication), but only if the constraint size is near 100 (less for preferential selection). At these constraint sizes, the non-specificity prevents the constraints from having an impact on table accuracy. Selecting 1000 constraints can cover more than 100% transaction value at sizes of less than 10 elements, and hence improve the AMAD significantly[16]. Above coverages of 1000% no more changes occur because of the high degree of duplication in the constraint information. Running more than 3000 low-sized constraints, the AMAD could be reduced to less than 30%.

4.4 Influence of the constraint structure

We now turn to variations of constraint structure, as illustrated in Fig. 1. Our first observation is that for single-element constraints (size = 1 in Fig. 9) the curves for dis-

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[16] A number of runs with fixed total constraint $ values of 10%U, 30%U and 50%U were carried out, showing that many small constraints performed better than a few large constraints.

Fig. 9: Comparison of balancing adjacent (A) and disjunct (D) structures, under equal (1) and preferential (2) selection (U^I initial estimate, total $-value undefined, 500 constraints, test on U). The diagram is organised as in Fig. 7.
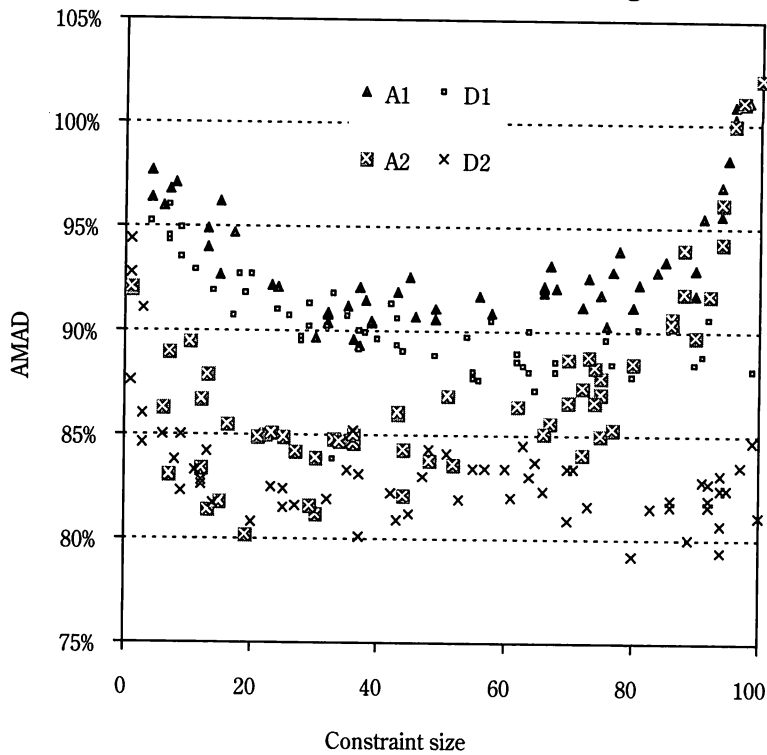

junct and adjacent structures coincide for equal (□▲) and preferential (×▨) selection. This is to be expected, since an adjacent structure of size 1 is essentially a disjunct structure. For constraint sizes of 30 and larger however, the adjacent constraint structures perform worse than the disjunct ones, and above size 90, their curves (▲▨) join. This deterioration is due to the fact that the constraint length approaches the dimension of the flow matrix, so that long constraints more or less coincide with already existing row and column sum constraints, thus effectively reducing the numbers of constraints through increasing duplication.

Increasing the focus on large elements (△+○ in Fig. 10) improves the performance of adjacent structures at low constraint sizes, but not at large sizes. For $r = 3$, a U-shaped form becomes apparent, which is due to two effects for declining performance: 1) small numbers of set elements at low sizes, and 2) overlap with row/column sum constraints at large sizes. In order to concentrate constraints even more on large flows, we selected seed elements from the ranking list, starting with the top element for the first constraint, and then proceeding down the list, with all other set elements attached in random directions, as before (labeled A∞). Once again, performance improves at low constraint sizes, but not at large sizes. In practice, however, it is unlikely that information can be found exclusively for top-ranking flows, so that in the following we will restrict ourselves to the preferential selections as described in Sec. 1.1.4.

Fig. 10: Performance of balancing adjacent (A) structures under equal (1) and preferential (2, 3, ∞) selection (U^I initial estimate, total \$-value undefined, 500 constraints, test on U). The diagram is organised as in Figs. 7 and 9.
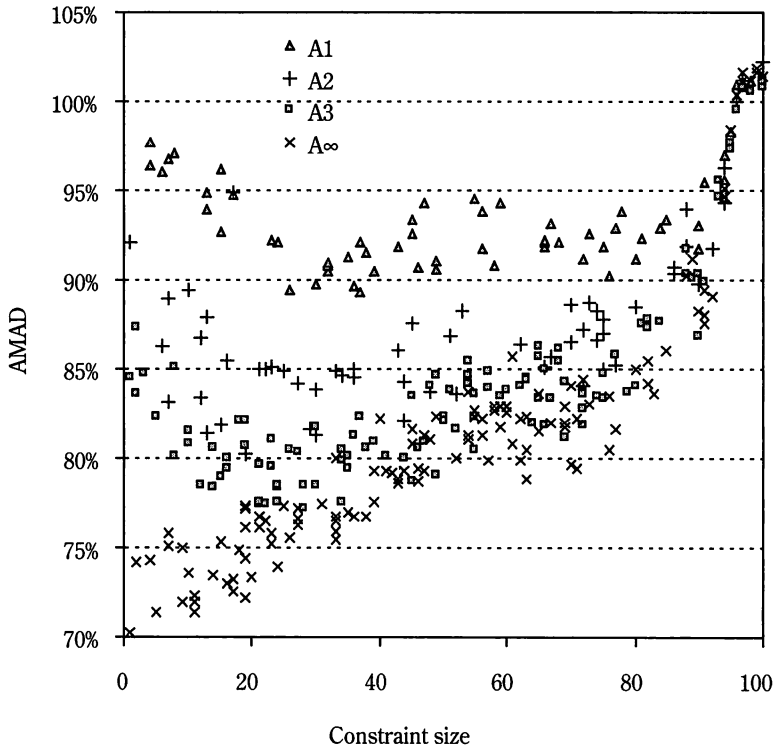

4.5 Influence of the accuracy type

We will now compare the balancing effects on the flow matrix, the Leontief inverse, and multipliers. It is well known that the uncertainty of a sum of many numbers can be considerable lower than the uncertainties of any of the numbers. This effect can be observed when the flow matrix U is converted into a Leontief inverse $L = (I - U\hat{x}^{-1})^{-1}$, and – further increasing the degree of summation – into output and CO_2 multipliers $\hat{\xi} = 1L$ and $\chi = cL$ (Fig. 11)[17]. The table accuracy of L in terms of the AMAD is considerably lower at around 30% than that of U, which is due to the stochastic error cancellations during the numerous additions and multiplications during the matrix inversion. However, multipliers (and not L) are likely to be the objects of practical applications. Thus aggregating even further, the (holistic) accuracy of output multipliers $\hat{\xi}$ is extremely low at around 2%. CO_2 multipliers χ are less accurate at around 20%, because in the case of Australia CO_2 intensities c are exceptionally high for only a few industries such as beef cattle grazing, beef products, aluminium or electricity generation, so that the overall uncertainty is largely determined by these few contributions, leading to a lower degree of error cancellation.

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[17] Probably first observed by Hewings (1977, p. 934).

Fig. 11: Performance of cRAS runs for the initial estimate U^I, N disjunct constraints of varying size and selectivity, and undefined $ value, tested *on the* Leontief inverse L (top), output multipliers ξ (middle), and CO_2 multipliers χ (bottom). The diagrams are organised as in Figs. 7, 9 and 10.

Average constraint size
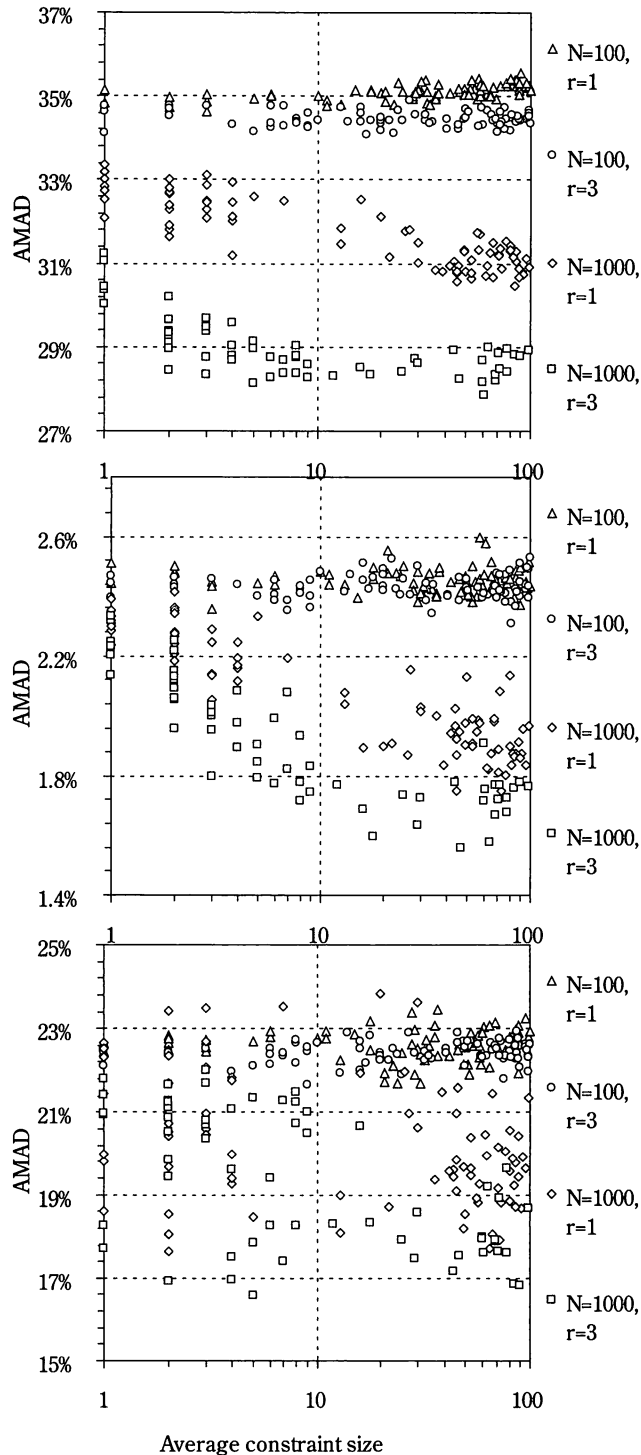
Tab. 1: AMAD of cRAS runs on 500 adjacent constraint structures of varying size and value, but with preferentially selected seed elements ($r = 2$), including its standard deviation over constraint sizes 1 to 100 (curve shapes as in Fig. 9), for different preliminary estimates and accuracy types.

Preliminary estimate \ Accuracy type	U	L	ξ	χ
U^R	97.1±5.3	33.5±1.1	2.0±0.3	23.7±1.9
U^I	85.9±9.5	30.8±3.3	1.8±0.3	22.4±2.9
U^{95}	20.3±1.0	7.2±0.3	0.4±0.04	3.0±0.6

We repeated the above numerical experiment for adjacent structures, and under variation of the initial estimate (Tab. 1). In summary, holistic accuracy is always higher than table accuracy (improving from U via L to ξ and χ), and updating (U^{95}) yields much better results than estimating (U^I and U^R), with the assumption of independence performing only slightly better than a random preliminary estimate (compare Sec. 1.2).

4.6 Sequence-(in)variance

As already shown for ordinary RAS by Bacharach 1970, in order to fulfil the minimum information gain property, the cRAS balance outcome has to be unique, and in particular independent of the order of the constraint sequence. A large number of cRAS runs of the adjacent, slightly selective type ($r=2$) were carried out, with constraint numbers between 50 and 500, and constraint sizes between 5 and 100, and tested on U. In all instances, the balancing result was unique.

4.7 Other issues

As with the Linear Programming and minimum information gain methods, (c) RAS preserves any preliminary zero values. Zero-value data in a preliminary matrix to be updated (for example U^{95}) may sometimes reflect technological impossibilities but – as Bacharach (1970, p. 26) pointed out – sometimes actually be an approximation of a very small positive value[18], in which case the zero should not be preserved. On the other hand, setting preliminary zero-value data to an arbitrarily small positive value may allow for positive updated values, but may also lead to non-zero transactions that in reality are technically impossible. We compared two runs of the adjacent, preferentially selected ($r = 2$) type, with one run starting with the original 1995 flow matrix U^{95}, and the other having all zero elements in U^{95} set to a small value. We obtained no discernible differences in the AMAD of the cRAS estimates. This is probably due to the fact that zero elements in U^{95} are either zero in U^{97}, or positive but small, and thus their contribution to the AMAD is negligible in any case.

In real-world applications, data for additional constraints may not actually supe-

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[18] Or, in the case of temporal forecasting, technical innovation may turn a truly zero input into a positive input.

rior, but even conflicting. Due to the nature of industry surveys proceeding as separate sales and purchasing (supply and demand) recordings, even corresponding row and column totals of gross output and input may be different. Even a small discrepancy between nominally equivalent constraints can prevent convergence of the (c)RAS method, most obviously if the given balancing threshold is smaller than the discrepancy. Reconciliation is therefore necessary, possibly by using a RAS technique on the conflicting constraints themselves, or by using Lagrangian or optimisation technique with uncertainty or reliability terms embedded in the function to be minimised[19].

5 Conclusions

We have described the performance of a RAS matrix balancing variant, called 'constrained RAS' or cRAS. In contrast to previous approaches for the estimation of input-output tables, cRAS allows incorporating constraints on arbitrary-sized and –shaped sets of elements into the preliminary estimate. These elements may be disjunct or adjacent, and exhibit a widely varying degree of overlap. We have tested cRAS's performance for the example of the 1996-97 Australian input-output flow matrix, using a number of distance measures.

First, the preliminary estimate is of crucial importance for the accuracy of the balancing outcome. Our experiments yielded table accuracies of balanced matrices of around 90% for estimated preliminaries, and around 20% for the previous-year preliminary. They showed that a preliminary based on some distribution assumption and row and column sums may even not be much better than a random matrix.

Second, including more constraints into the balancing process always improves the balance outcome. This is especially the case if constraints are small, numerous, disjunct, and concentrated on important matrix cells. This result agrees with previous findings by Jensen and West 1980, in the sense that for many analytical purposes, it is more important to estimate a few strategic table elements as accurately as possible, than to estimate as many elements as possible. In this context, we also found it important to "pin-point" such strategic elements in disjunct, small constraints, rather than have the value of say one important element "buried" in an adjacent, large constraint. While the size of a constraint directly diminishes the ability to uncover one particular strategic element from its summed value, an adjacent structure means that this uncertainty affects the industry sector to which the strategic element belongs.

Third, the accuracy of derived quantities (such as inverse matrices and aggregates thereof) is always higher than the accuracy of the flow matrix itself. This is due to cancellation of stochastic error components during numerous additions and multiplications that lead to the derived quantities. Holistic accuracy – as this effect has been termed –

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[19] A number of authors discuss balancing techniques that include uncertainty and reliability, using either RAS (Jensen and McGaurr 1976; 1977; Lahiri 1984), variance minimisation (Gerking 1976; 1979a; b; Miernyk 1979), or Lagrangian minimization (van der Ploeg 1982). All methods have in common the problem that probabilities for conflicting data are mostly unavailable, so that subjective judgment has to be consulted.

is therefore certainly a concept to be kept in mind when compiling and using inter-industry tables. Finally, as ordinary RAS, cRAS produces a unique balancing outcome.

Given the scarcity of superior data, especially when estimating regional or physical input-output tables, the main value of cRAS is that the analyst can make use of any kind of information they can find, irrespective of the degree of aggregation.

Acknowledgements

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